Is Sustainable Development Compatible with Rawlsian Justice?

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Abstract

Does economic justice stymie economic development? This paper demonstrates that sustainability is compatible with Rawlsian intertemporal justice, even when considering human capital and natural resources. The methodology employed herein extends and amends previous works that (1) do not consider human capital or renewable resources, and (2) rely upon the application of standard Lagrangian methodology to a continuum of nonlinear constraints. This approach circumvents problems associated with earlier works by internalizing constraints and demonstrating two sufficient conditions which guarantee existence of a Rawlsian maximin path.

KEYWORDS: endogenous growth, sustainability, Rawlsian maximin, intertemporal justice, renewable resources, nonrenewable resources

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1. Introduction

This analysis demonstrates the existence of a Rawlsian (1971) maximin solution to the problem of endogenous sustainable growth in a model which includes both the accumulation of human capital and heterogeneous natural resources. These results are then applied to a model of endogenous growth involving natural resources and the accumulation of human capital. Solow (1992) postulates that sustainability "depends on a bundle of endowments, in principle on everything that could limit the economy's capacity to produce economic well-being. That includes nonrenewable resources of course, but it also includes the stock of plant and equipment, the inventory of technological knowledge, and even the general level of education and supply of skills." Cass and Mitra (1991) and Faucheux, Muir and O’Connor (1997) recognize the diverse attributes of various inputs. While inputs may be technical complements, as in Hartwick (1977), their diverse and unique attributes often require dynamic substitutability. Ultimately, sustainability depends on initial endowments, available technology and the intertemporal objectives of society, or equivalently, a benevolent central planner. Therefore, the model of intertemporal growth presented herein incorporates physical capital and human capital with both renewable and nonrenewable resources. Production is assumed to be concave and homogeneous. The objective is Rawls' maximin principle of intergenerational justice. Kaganovich (2000), Murrel (1980), and Phelps and Riley (1978) define a Rawlsian maximin growth path to be one that maximizes the welfare of the least fortunate generation. That is, over all possible growth paths, the maximin path is the one that offers the greatest minimum level of consumption over time.

Ramsey, Arrow (1973), Dasgupta (1974), and Page (1977, 1997) offer justification for the pursuit of non-market allocation mechanisms such as the maximin. Page (1997) states, “Markets can be expected to allocate resources more or less efficiently relative to a given distribution of wealth or market power. But markets cannot be expected to solve the problem of what is a fair or equitable distribution of wealth, either among different people at a point in time (intratemporally) or among different generations (intertemporally).” According to Ramsey (1928, p.543), "(discounting) later enjoyments in comparison with earlier ones (is) a practice which is ethically indefensible and arises merely from the weakness of the imagination." For these reasons Solow (1974), Stiglitz (1974), Asako (1980) and others have offered models featuring intertemporal paths of constant per capita consumption.

The unifying theme of the previous generation of models was the characterization of "regular" maximin paths. These paths are such that the maximin solution is one in which per capita consumption is constant throughout time. In particular, Dasgupta and Heal (1979), Dixit, Hammond and Hoel (1980),
and Becker (1982) present continuous time models with maximin paths that are indeed regular, provided the maximin path exists. Thus, to ensure their own conclusions, each assumes the existence results of Burmeister and Hammond (1977).

Using a continuous-time model of heterogeneous capital, Burmeister and Hammond set out to prove the existence of a maximin path. Then, assuming the existence of a regular maximin path they characterize such a path, should it exist. The method they choose to demonstrate existence of the maximin relies on the ability to use the Lagrangian method in the presence of a continuum of multipliers. In fact, how to correctly and effectively handle a continuum of nonlinear constraints remains an open mathematical question. For example, Hager (1990) derives error estimates for an augmented Lagrangian approximation to the continuous-time optimal control problem. In another work, Polak, Yang and Mayne (1993) use a method of "centers" based on barrier functions for solving an specific optimal control problem with a continuum of constraints. Thus, although Dasgupta and Mitra (1983) demonstrate the existence of a regular maximin path in a discrete-time model with a general production function, existence of a maximin solution to a model of economic growth in continuous time remains an open question. More significantly, Dixit, Hammond and Hoel (1980) and others have explicitly assumed the Burmeister and Hammonds' proof of the existence of a maximin solution when characterizing regular maximin paths in the presence of a nonrenewable resource. In a similar fashion, Becker (1982), uses the Burmeister-Hammond result when investigating the relationship between the accumulation of physical capital and erosion of an environmental good. Thus, one function of this paper is to provide justification for the existence of the maximin in continuous-time models of economic growth.

This discussion modifies the method employed by Becker, Boyd and Sung (1989) to accommodate the specific nature of the maximin objective function as well as the inclusion of human capital. One finds that existence of an optimal solution depends on showing that the feasible set is compact and the felicity is upper semicontinuous. Subsequently, this result is applied to a model with human capital and either nonrenewable or renewable resources. The primary advantages of this approach are, (i) it avoids the potential pitfalls associated with managing a continuum of dynamic constraints and (ii) it is possible to apply this existence result to a model which includes an increasing rate of population growth, the accumulation of human capital, and both nonrenewable and renewable resources.
In section two a model of endogenous growth with human capital is demonstrated so as to provide an impetus for the application of the maximin objective. Section three proves sufficient conditions for the existence of a maximin solution. Section four demonstrates that the general existence result can be applied to a model involving either renewable or nonrenewable resources. Section five concludes the paper.

2. A Simple Model of Endogenous Sustainable Growth with Heterogeneous Resources

This model is a synthesis of models presented by Lucas (1988) and Solow (1956, 1974). This particular endogenous growth mechanism was first postulated by Uzawa (1965) and later extended by Lucas. One could say that this approach differs from the Romer (1990) approach in that it concentrates on the source through which new technology is born, human knowledge, as opposed to the medium through which it is acquired, the market. The Lucas model incorporates both physical and human capital constraints within a dynamic framework. Growth depends directly on human capital formulation and the rate which human capital develops essentially depends on a desire to forego some measure of current consumption. Thus, in contrast to Romer (1990), the nature of growth paths depends on some measure of intergenerational benevolence. Thus, it would seem that the Lucas model is more ideally suited for a discussion of intergenerational equity within the confines of an endogenous growth model. The mechanism for endogenous growth consists of a control variable governing the rate at which human capital is enhanced. The optimal path of human capital accumulation is one that satisfies the maximin criterion.

To be specific, first define \( N(t) = N_0 e^{nt} \) to be the population at time \( t \). Population grows at a rate, \( n \). Assume full employment. In the spirit of Lucas, at each interval of time the \( i^{th} \) individual has the option to choose the percentage, \( \mu_i \), of his productive time that he wishes to devote toward working in a productive capacity. The remaining time, \( 1 - \mu_i \), is devoted to the development of human capital. That is, the individual engages in some activity which increases his on the job productivity. One may envision this endeavor as some sort of scholarly activity or job training. Human capital is interpreted as skill or expertise resulting from off-the-job training, as opposed to Arrow's (1962) 'learning by doing' interpretation. Existing human capital, \( h_i \), is then incorporated into the labor component. This effectively creates a labor component that is dependent upon both the size of the population and the existing level of skill.
Let human capital evolve over time according to the equation,
\[ \dot{h}_i^t = \delta^i h_i^t (1 - \mu_i^t). \]
The coefficient, \( \delta^i \), indicates how efficiently an individual is able to develop his human capital. Next, define effective labor as
\[ L_t = \int_{1}^{N(t)} \gamma_t(i) \mu_t(i) h_t(i) \, di, \]
where \( \mu_t(i) = \mu_i^t \) corresponds to the hours the \( i^{th} \) individual devotes to work during time period \( t \), \( h_t^i \) represents the skill level of the \( i^{th} \) individual at time \( t \), and \( \gamma^i \) is a coefficient indicating the ability of the \( i^{th} \) individual to convert skill-hours into labor. Thus, human capital is accumulated by devoting potentially productive time to learning. This increases the value of the human capital component and, as a result, effective labor. In what follows, effective labor and physical capital (interpreted, for example, as machines or equipment) are inputs to a production function that yields a single good which can be consumed or used to augment physical capital.

For tractability, Lucas' assumption that the individuals of this economy are identical is adopted. That is, they possess identical skill levels and they choose their productive hours similarly. Also, without much loss of generality, assume that the efficiency parameters, \( \gamma \) and \( \delta \), are constant. Allowing \( \gamma \) and \( \delta \) to vary over time merely adds an exogenous component to the growth mechanism of the economy. Once this simplification is made, effective labor becomes
\[ L_t = N_t \gamma \mu_t h_t = N_t \gamma \mu_t h_t. \]
Thus, effective labor depends on the development of human capital as well as the size of the population.

The production function is \( F(L_t, K_t, Q_t, R_t) \), where \( L_t \) is the labor/human capital component, \( K_t \) is physical capital, \( Q_t \) is the rate at which a nonrenewable resource is extracted, and \( R_t \) is the harvest rate of a renewable resource. When considering a nonrenewable resource, \( Q_t \) can not exceed \( W_t \), where \( W_t \) represents the stock of the resource at time \( t \). If the nonrenewable resource is to be sustained over an infinite horizon, the constraint is then
\[ \int_{t=0}^{\infty} Q_t \, dt \leq W_0. \]

\[ ^1 \text{For reasons that become clear later, the model is restricted to one physical capital/consumption good. However, the vector of inputs can be trivially extended to include multiple nonrenewable and renewable resources.} \]
The variable, $R_t$, represents the harvest rate of the renewable resource. Associated with any renewable resource is a spawning function, $S(X_t)$, where $X_t$ represents the current renewable resource stock or population. Thus, changes to the renewable resource stock are defined by $X_t = S(X_t) - R_t$.

It is assumed that the spawning rate satisfies the principles of logistic growth (Clark (1990), Clark, Clarke and Munro (1979)). That is, there exists an interval, $X_{\text{max}} \leq X_t \leq X_{\text{max}}$, where initially, $S'(X_t) \geq 0$. As the population grows, $S(X_t) \geq 0$, increased competition for food and living space eventually causes the spawning rate to fall, $S'(X_t) < 0$. The spawning function is concave, $S''(X_t) \leq 0$, throughout the interval and $X_{\text{max}}$ represents the maximum sustainable population level. It follows that $X_{\text{max}}$ is also stable. Therefore, at any given moment in time, $R_t$ is bounded above by $X_{\text{max}}$.

Assume that all inputs are necessary for production, so $0 < \mu_t \leq 1$. Let demand for output be $Y_t$ and assume that aggregate demand equals aggregate output, $Y_t = F(L_t, K_t, Q_t, R_t)$. Also assume that the production function $F(\cdot)$ is increasing, concave, and homogeneous of degree one with respect to both factors of production. Production is defined by $Y_t = C_t + (\dot{K}_t + \zeta K_t)$, where $C_t$ is the portion of capital the economy chooses to consume at time $t$. Gross investment is $I_t = \dot{K}_t + \zeta K_t$, where $\zeta$ is the rate at which existing capital depreciates. Since the production function is HD(1), it can be rewritten as $Y_t = L_t f \left( k_t, q_t, r_t \right)$, where

\[
    k_t = \frac{K_t}{L_t}, \quad q_t = \frac{Q_t}{L_t}, \quad r_t = \frac{R_t}{L_t}, \quad \text{and} \quad f(k_t, q_t, r_t) = F \left( 1, \frac{K_t}{L_t}, \frac{Q_t}{L_t}, \frac{R_t}{L_t} \right).
\]

This allows one to express the remaining inputs in terms of effective labor. Define "effective" physical capital to be physical capital relative to effective labor. Define other variables similarly. The properties of $F(\cdot)$ ensure that the transformed production function, $f(k_t, q_t, r_t)$, is increasing and concave in each input.

Letting $y_t = \frac{Y_t}{L_t}$, $i_t = \frac{I_t}{L_t}$, and $c_t = \frac{C_t}{L_t}$, it follows that

\[
y_t = f(k_t, q_t, r_t) = c_t + i_t = c_t + \left[ \frac{\dot{K}_t}{L_t} + \zeta k_t \right].
\]

Differentiating $k_t$ one finds that, $\dot{k}_t = \frac{\dot{K}_t}{L_t} - \frac{\dot{L}_t}{L_t} k_t$. Next, substitute for $\frac{\dot{K}_t}{L_t}$ to obtain $y_t = c_t + \left[ \frac{\dot{L}_t}{L_t} k_t + \dot{k}_t + \zeta k_t \right]$. 

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The percentage change in effective labor at each period of time is \( \hat{L}_t = \frac{n \gamma N_t \mu \hat{h}_t + \gamma N_t \mu \hat{h}_t}{\gamma N_t \mu \hat{h}_t} \), or \( \hat{L}_t = n + \frac{\hat{h}_t + \hat{\mu}_t}{\mu_t} = n + \delta (1 - \mu_t) + \hat{\mu}_t. \)

Substituting this expression into the equation for \( y_t = f(k_t, q_t, r_t) \) and rearranging, investment is

\[
\dot{k}_t = f(k_t, q_t, r_t) - c_t - \theta_{\mu} k_t,
\]

where \( \theta_{\mu} = n + \delta (1 - \mu_t) + \hat{\mu}_t + \zeta \), or

\[
\theta_{\mu} = n + \zeta + \hat{h}_t.
\]

The expression, \( \delta (1 - \mu_t) + \hat{\mu}_t \), indicates how effective physical capital, \( k_t \), changes as labor is enhanced by the development of human capital. By expression (2), the capital modification rate, \( \theta_{\mu} \), depends on the rate at which the population grows, the rate at which human capital accumulates and the rate at which physical capital depreciates (often forsaken for tractability). In contrast to Lucas (1988), this analysis does not necessarily assume that growth is balanced (\( \hat{\mu}_t \) may be non-zero). The model is endogenous in that the optimal path of human capital is that which satisfies the maximin objective function. Specifically, the results that follow ensures that there exists a path \( \mu_t, t \geq 0 \), which guarantees the existence of feasible paths, \( L_t, K_t, \dot{K}_t, Q_t, R_t \), and \( C_t \), consistent with a maximin solution. In particular, the central planner must determine that there exist feasible paths ensuring the existence of

\[
\sup_{c, k} \left\{ \inf_{t} u(t, k, \dot{k}, c) \right\},
\]

subject to (1), where it is assumed that the real valued utility function (felicity), \( u(\cdot) \), is jointly continuous in \( t, k, \dot{k}, \) and \( c. \)

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2It is tempting, and perhaps intuitive, to refer to this as effective investment. However, using the term, "effective," in this manner is not consistent with the previous definition.

3Lucas (1988) chooses \( \mu_t \) and \( c_t \) so as to maximize an intertemporal objective involving CES preferences over consumption in a deterministic model of optimal control as defined by Kamien and Schwartz (1991).
3. Existence of a Maximin Solution to the Sustainable Development Problem

In this section existence of the maximin is proved for the general case where there are $m$ distinct inputs. In order to emphasize the generality of the following argument, in this section the notation is simplified. Let $k = (k_1, k_2, \ldots, k_m) \in \mathbb{R}^m$ now denote a vector of heterogeneous (physical capital, human capital, resource) inputs. For example, when applied to the model presented in section two, $k$ becomes the input vector $(k_t, q_t, r_t) \in \mathbb{R}^3$.

The maximin objective is

$$v = \max_{c,k} \left\{ \min_i u(t, c, k, \dot{k}) \right\},$$

or

$$v(c, k) = \inf_i u(t, c, k, \dot{k}).$$

Our notation has been modified for this discussion. Note that for the proof of existence of a maximin solution, $k(t)$ is now the stock of physical capital at time $t$, and $\dot{k}(t)$ denotes the change in inputs at time $t$, including investment in (effective) physical capital. Finally, $c(t)$ is consumption at time $t$. Moreover, by equation (1), the consumption path is uniquely defined by $t, k$, and $\dot{k}$. Thus, consumption need not be an explicit argument of the felicity (Becker, Boyd and Sung (1989)).

The technique for proving existence is provided by Becker, Boyd and Sung (1989), cited hereafter as BB&S. However, because BB&S define preferences in terms of a recursive, discounted utility function, the methodology must be modified to suit maximin preferences. As in BB&S, existence of an optimal solution depends on showing that the feasible set is compact and the felicity is upper semicontinuous. The primary distinction here is that upper semicontinuity of the maximin felicity does not follow from their analysis involving recursive utility. However, the compactness argument found in BB&S does directly apply to this analysis. In order to balance completeness with brevity, the compactness argument is summarized. The main purpose of this section is to prove upper semicontinuity of the maximin felicity.

We assume that there is a vector of inputs, $k$, corresponding to an output vector, $y \in \mathbb{R}^m$. The feasible technology, is a measurable set, $\Omega \subset \mathbb{R}^{1+2m}$. It is assumed that the technology set at time $t$, $\Omega_t = \{(k, y) : (t, k, y) \in \Omega \}$, is non-empty for every $t$, and that $\{k : (t, k, y) \in \Omega, y \in \mathbb{R}^m \}$ is closed for every $t \geq 0$. As noted in Becker and Boyd (1992), these conditions must also be assumed in Becker, Boyd and Sung (1989).
generalized production function is \( G(t,k) = \{ y : (t,k,y) \in \Omega \} \). The feasible output domain is then \( D = \{ (t,k) : \exists y \text{ where } (t,k,y) \in \Omega \} \). As a result,

\[
D(t_0) = \{ k : \exists y \text{ where } (t_0,k,y) \in \Omega \}
\]

represents the set of all feasible programs for any time \( t_0 \). Given the capital stock at time \( t \), any portion of the output, \( y \in G(t,k) \), can be consumed or devoted to the accumulation of physical capital. As in BB&S, it is assumed that the capital accumulation program is an absolutely continuous function, \( k : \mathbb{R} \rightarrow \mathbb{R}^m \). Let the set of absolutely continuous and feasible programs be denoted by \( A \). This assumption allows \( k(t) \) to be written as the integral of its derivative, \( \dot{k}(t) \). That is,

\[
k(t) = k(0) + \int_0^t \dot{k}(s) \, ds.
\]

Absolute continuity of \( k \) also ensures that \( \dot{k} \) exists almost everywhere and is locally integrable.\(^5\)

Define the set of admissible programs from any given initial stock, \( x \in X \), by

\[
U(x) = \{ k \in A : \dot{k} \in G(t,k) \text{ a.e.}, 0 \leq k(0) \leq x \}.
\]

The technology is convex if the \( t \)-section \( \{ (k,y) : y \in G(t,k) \} \) is convex for each \( t \). It can be shown that \( U(x) \) is a convex set with respect to each \( x \in X \).

It is assumed that preferences depend on a continuous, concave felicity, \( u \). In addition, since we assume that \( k \) is defined almost everywhere (a.e.), \( \inf \) is actually the essential infimum. The essential infimum of \( u \) is \( \sup \{ \inf \limits_t \{ w(t,k,k) \} : w \text{ any member of the equivalence class of } \}

\[
\text{where } w \text{ any member of the equivalence class of functions equal to } u \text{ almost everywhere.}
\]

The maximin objective functional, \( v \), is then

\[
v(k) = \text{ess inf} u(t,k,k).
\]

The value function is defined by

\[
J(x) = \sup \{ v(k) : k \in \mathcal{A}(\mathcal{O}) \}.
\]

An admissible program, \( k^* \in U(x) \), is considered optimal if \( v(k^*) = J(x) \).

\(^5\)Local integrability insures that integration over a compact set is finite. This property is denoted by \( k \in L^1_{loc} \).
3.1. **Compactness of the Feasible Set**

Three constraints on the technology must be satisfied in order to ensure existence of an optimal program.

(i) $G(t, k)$ must be compact and convex for each $(t,k)$ and $k \rightarrow G(t,k)$ must be upper semicontinuous for each $t$.

(ii) For every initial stock of physical capital, $x \in X$, there must exist a locally integrable function $\eta_x \in L^1_{loc}$, such that $|k| \leq \eta_x$ almost everywhere whenever $k \in U(x)$.

(iii) $U(0) \neq \emptyset$.

BB&S show that $U(x)$ is a compact subset of $A$ when these technology conditions are satisfied. In short, when combined with the upper semicontinuity of the objective function, this result ensures existence of the maximin. In order to prove that an optimal solution exists it is necessary to define two distinct topologies for the function space, $A$. This issue arises because the compact-open topology is best suited for the input stocks whereas the weak topology naturally arises from the input flows, i.e. investment and changes in extraction or harvest rates (BB&S (p. 82)). To summarize, Becker Boyd and Sung demonstrate that these two topologies are equivalent on $A$. That is, if a sequence converges in $A$ with respect to one topology, then it necessarily converges with respect to the other topology as well. The topology of a set defines its open sets. Thus, it may also be said that two topologies are equivalent if whenever a set is open with respect to one topology, it is also open with respect to the other.

The first topology considered is the compact-open topology. The compact-open topology is defined by the family of norms, $|k|_{0,T} = \sup_{0 \leq t \leq T} |k(t)|$, where $|\cdot|$ is any of the equivalent $\ell^p$ norms on $\mathbb{R}^m$. Under this topology, $A$ is a complete, countably normed (Fréchet) space. This topology is used to insure compactness of the feasible set, $U(x)$. The second topology to be considered is the weak topology defined on $A$. This topology is used to insure that the objective is upper semicontinuous. In order to properly define a weak topology on $A$, BB&S first demonstrate that $A$ is the direct sum $\mathbb{R} \oplus L^1_{loc}$. This is accomplished by defining a norm on $A$ which uses the fact that the derivatives (input flows) of absolutely continuous functions (capital stock) are locally integrable, and that absolutely

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$6$Recall that $L^p = \left\{ k : \|k\|_p = \left( \sum_{i=1}^{\infty} |\alpha_i|^p \right)^{\frac{1}{p}} < \infty \right\}$. See Packel (1974, p.55).
continuous functions can be described as the integral of its derivative. BB&S then show that the topology defined on $A$ with respect to this norm is equivalent to the topology on $\mathbb{R} \oplus L^1_{\text{loc}}$ under the natural metric. Thus, $A = \mathbb{R} \oplus L^1_{\text{loc}}$. This makes $A$ a Fréchet space, ensuring the existence of a dual space. The weak topology is defined by the dual space, the space of linear functionals on $A$. By definition, the weak topology on $A$ is the weakest topology which ensures that each linear functional of $A$ is continuous. In order to ensure compactness of the feasible set, $U(x)$, and upper semicontinuity of the objective simultaneously, it is essential that the two topologies be equivalent. This is achieved by focusing on the following subspace of $A$, $F(\eta) = \{ k \in A : |k| \leq \eta \}$ where $\eta \in L^1_{\text{loc}}$. As a result, the two topologies are equivalent on the subspace consisting of those programs where the accumulation of physical capital is bounded by some $\eta \in L^1_{\text{loc}}$. The compactness results from BB&S are summarized here.

**Result 1:** On $F(\eta)$, the compact-open and weak topologies are equivalent. (i) $F(\eta)$ is compact in the compact-open topology. Moreover, $F(\eta)$ is metrizable. (ii) $F(\eta)$ is complete under the uniform metric.

**Result 2:** If the technology conditions are satisfied, $U(x) \subset F(\eta)$ is a compact subset of $A$.

### 3.2. Upper Semicontinuity of the Maximin Function

In order to show that the maximin objective is upper semicontinuous it is first necessary to generate some more general preliminary results concerning upper semicontinuity.

**Definition:** Let $f$ be a real valued function defined on a subset $E$ of a metrizable topological space, $X$. Then $f$ is said to be upper semicontinuous at $k \in E$ if given $\varepsilon > 0$, there exists $\delta > 0$ such that $f(x) < f(k) + \varepsilon$ for all $x \in B(k; \delta) \cap E$. A function is upper semicontinuous on $E$ if it is so for every $k$ (Hasser and Sullivan (1991)).

It will be more convenient to use an alternative means of identifying upper semicontinuity. Once this alternative definition is justified, it will be helpful to investigate a particular property of convergent sequences of upper semicontinuous functions. Note that convergence is predicated on the weak topology.
**Proposition 1:** The functional, $f$, from a subset, $E$, of a metrizable topological space, $X$, is upper semicontinuous at a point $k \in E$ if and only if

$$f(k) = \lim_{\delta \to 0^+} \sup \{ f(x) : x \in B(k; \delta) \cap E \}.$$ 

**Proof:** ($\Rightarrow$) Assume $f$ is upper semicontinuous at $k \in E$. Let $\varepsilon > 0$ be given. Since $f(x) < f(k) + \varepsilon$ for all $x \in B(k; \delta) \cap E$, it follows that

$$\sup_{\delta > 0} \{ f(x) : x \in B(k; \delta) \cap E \} \leq f(k) + \varepsilon,$$

so,

$$\inf_{\delta > 0} \{ \sup_{\delta > 0} \{ f(x) : x \in B(k; \delta) \cap E \} \} \leq f(k).$$

Since $x \in B(k; \delta) \cap E$ for all $\delta > 0$, it follows that

$$f(k) = \inf_{\delta > 0} \{ \sup_{\delta > 0} \{ f(x) : x \in B(k; \delta) \cap E \} \}.$$

Also, since $\sup_{\delta > 0} \{ f(x) : x \in B(k; \delta) \cap E \}$ is non increasing as $\delta$ approaches $0^+$,

$$\inf_{\delta > 0} \{ \sup_{\delta > 0} \{ f(x) : x \in B(k; \delta) \cap E \} \} = \lim_{\delta \to 0^+} \{ f(x) : x \in B(k; \delta) \cap E \}.$$ 

($\Leftarrow$) Next, assume that

$$f(k) = \lim_{\delta \to 0^+} \{ f(x) : x \in B(k; \delta) \cap E \}.$$ 

Let $\varepsilon > 0$ be given. If $f$ is not upper semicontinuous, for every $\delta > 0$ there exists some $x_\delta \in B(k; \delta) \cap E$ such that $f(x_\delta) > f(k) + \varepsilon$.

Then,

$$\lim_{\delta \to 0^+} f(x_\delta) \geq \lim_{\delta \to 0^+} f(k) + \varepsilon.$$

But,

$$\lim_{\delta \to 0^+} \{ f(x) : x \in B(k; \delta) \cap E \} = f(k),$$

so,

$$f(k) \geq \lim_{\delta \to 0^+} f(k) + \varepsilon = f(k) + \varepsilon.$$

This is an obvious contradiction. Thus, $f(\cdot)$ must be upper semicontinuous.

*Q.E.D.*
Proposition 2: Let \{f_\alpha\}, \alpha \in G, be a family of upper semicontinuous functions on a subset, E, of a metrizable topological space X. If there exists a sequence of functions \{f_n\}, n \in G, such that \{f_n\} decreases monotonically to a function, f, then f is also upper semicontinuous.

Proof: Let k be an arbitrary element of \(E \subset X\). Since each function of the sequence \{f_n\} is assumed to be upper semicontinuous, it follows that given \(\varepsilon > 0\), for each n there exists \(\delta_n\) such that if \(x \in B(k; \delta_n) \cap E\), then
\[ f_n(x) < f_n(k) + 2\varepsilon. \]

Also, since the sequence \{f_n\} converges to f monotonically from above, \(\inf f_n(x) = f(x)\), so \(f_n(x) \geq f(x)\) for every n and all \(x \in E\). This convergence also ensures the existence of some \(N_\varepsilon\) such that for every \(n \geq N_\varepsilon\),
\[ f_n(k) - f(k) < \varepsilon. \]

So, if \(x \in B(k; \delta_{N_\varepsilon}) \cap E\), then
\[ f_{N_\varepsilon}(x) < f_{N_\varepsilon}(k) + 2\varepsilon \]
with
\[ f_{N_\varepsilon}(k) - f(k) < \varepsilon. \]

Next, set \(\delta = \delta_{N_\varepsilon}\). Therefore, if \(x \in B(k; \delta) \cap E\), it then follows that \(f(x) < f(k) + \varepsilon\) and f is upper semicontinuous.

Q.E.D.

Recall that the goal is to show that \(\nu(k) = \inf_t u(t, k, k)\) is upper semicontinuous with respect to k in the weak topology. Once this is shown, existence of a solution to the intergenerational equity problem is assured. Application of Proposition 2 requires a well-defined metric for the set of admissible programs. However, recall that, on \(U(x)\), the compact-open topology is indeed metrizable.

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\(^7\)The precise conditions for metrizability require that the space be locally convex with a countable neighborhood basis about the origin. Given the weak topology, apply the usual supnorm to all the compact subsets of the topological space. This yields a family of continuous seminorms. (This is how the compact-open topology is defined.) This is sufficient to guarantee that the topological space is locally convex. Existence of a basis about the origin follows directly. Metrizability is confirmed by Packel (1974, p.15, 39, 43-45) and implemented by BB&S (1989, p.84).
Continuing, define
\[ v_{\tau_0}(k) = \inf_{t \in T_0} u(t, k, \dot{k}). \]

The first step is to show that \( v_{\tau_0}(k) \) is upper semicontinuous. Upper semicontinuity of \( v(k) \) will follow from the fact that \( v_{\tau_0}(k) \) converges monotonically to \( v(k) \) over time.

**Proposition 3:** Assume that the felicity, \( u \), is jointly continuous with respect to \( k, \dot{k}, \) and \( t \). Then, \( v_{\tau_0}(k) = \inf_{t \in T_0} u(t, k, \dot{k}) \) is upper semicontinuous on \( \mathcal{U}(x) \).

**Proof:** Let \( k_n, k \in \mathcal{A} \) for all \( n \). Note that as \( \delta \to 0^+ \),
\[
\sup_{k_n} \left\{ v_{\tau_0}(k_n) : |k_n - k| < \delta \right\}
\]
is non increasing. In fact, by letting \( \delta \to 0^+ \), this forces \( k_n \to k \) with respect to the weak topology on \( \mathcal{A} \). Since the two topologies are equivalent on \( \mathcal{F}(\eta) \subset \mathcal{A} \), \( k_n \to k \) in the compact open topology on \( \mathcal{F}(\eta) \). However, \( \mathcal{U}(x) \) is a compact subset of \( \mathcal{F}(\eta) \), so it follows that \( k_n \to k \) uniformly on \( \mathcal{U}(x) \) (Packel (1974, p.15)). For simplicity, define \( u(k_n) \equiv u(t, k_n, \dot{k}_n) \), etc.. Therefore, since \( u(\cdot) \) is continuous in \( k \), it follows that \( u(k_n) \to u(k) \). This implies that
\[
\inf_{t \in T_0} u(k_n) \to \inf_{t \in T_0} u(k),
\]
forcing
\[
\sup_{k_n} \left\{ \inf_{t \in T_0} u(k_n) \right\} \to \sup_{k_n} \left\{ \inf_{t \in T_0} u(k) \right\} = \inf_{t \in T_0} u(k).
\]
Thus,
\[
\lim_{\delta \to 0} \sup_{k_n} \left\{ v_{\tau_0}(k_n) : |k_n - k| < \delta \right\} = \inf_{t \in T_0} u(k) = v_{\tau_0}(k).
\]
Therefore, by propositions 1 and 2, \( v_{\tau_0}(k) \) is upper semicontinuous on \( \mathcal{U}(x) \).

Q.E.D.
Proposition 4: \( v(k) = \inf_t u(t, k, \hat{k}) \) is upper semicontinuous on \( U(x) \).

Proof: Note that if \( T_0 < T_n \), then, given any \( k \in U(x) \),
\[
\nu_{T_0}(k) \geq \nu_{T_n}(k).
\]
Thus, as \( n \to \infty \), \( \nu_{T_n}(k) \) is monotonically decreasing. In fact, continuity of the felicity ensures that
\[
\nu_{T_n}(k) \to v(k).
\]
Since each \( \nu_{T_n}(k) \) is upper semicontinuous, propositions 1 and 2 ensure that \( v(k) \)
is upper semicontinuous on \( U(x) \).
Q.E.D.

The following consequence of upper semicontinuity is necessary to prove existence of the maximin.

Proposition 5: Let \( k_n \in A \) and let \( k_n \to k \) weakly. If \( v(k) \) is upper semicontinuous then,
\[
\limsup_{n \to \infty} v(k_n) \leq v(k).
\]

Proof: Proceed by contradiction. If not, then
\[
\limsup_{n \to \infty} v(k_n(t)) > v(k(t)).
\]
Thus, at each \( t \) there exists \( \varepsilon_0 \) such that given any \( \varepsilon < \varepsilon_0 \), and for every \( \delta > 0 \),
there exists some \( k_{n_\delta}(t) \in B(k; \delta) \cap A \) with
\[
v(k_{n_\delta}(t)) > v(k(t)) + \varepsilon.
\]
This contradicts the definition of upper semicontinuity.8
Q.E.D.

8The converse is also true, although it is not necessary for this discussion.
Theorem 1: If the technology conditions hold and the objective function, \( v(k) \) is upper semicontinuous, then the maximin solution exists.

Proof: Since the technology conditions hold, \( \mathcal{U}(x) \) is compact. Let \( \{ k_n \} \) with \( k_n \in \mathcal{U}(x) \) be such that \( v(k_n) \) continually increases. Compactness of the feasible set ensures that there exists a convergent subsequence \( \{ k_m \} \) where \( k_m \to k^* \) with \( k^* \in \mathcal{U}(x) \). Since the maximin objective is upper semicontinuous, it follows that

\[
J(x) = \lim_{m \to \infty} \sup v(k_m) \leq v(k^*).
\]

But, since \( k^* \in \mathcal{U}(x) \), it also follows that \( v(k^*) \leq J(x) \). So, \( v(k^*) = J(x) \) with \( k^* \) as the maximin path.

Q.E.D.

All that remains is to show that the model from section two satisfies the technology conditions. Once this is demonstrated, existence of the maximin solution to the human capital accumulation model will follow directly from the results in section three.

4: Existence of a Maximin Solution to the Problem of Sustainable Growth

To ensure existence, one must verify that the necessary conditions for Theorem 1 continue to be satisfied when considering nonrenewable and renewable resources. The first and third technology conditions are not affected by the addition of a nonrenewable resource. With the addition of a third input, only the second technology condition is of concern. In this situation, as before, of primary concern is the growth rate of effective inputs, whether physical or environmental in nature. The immediate concern is to show that the paths denoting the rates of change associated with each input are bounded by a locally integrable function. By definition, the initial stock of a nonrenewable resource is its upper bound. Thus, it follows that the second technology condition is satisfied by the basic property of a nonrenewable resource (1991). Also recall from section two that the harvest rate for a renewable resource is bounded above by its maximum

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9This result follows as in BB&S (1989, p.90), albeit for the maximin felicity.
10In fact, BB&S present their argument in terms of several capital goods.
sustainable population level. In order to capture specific attributes we now revert
to the more cumbersome notation of section two.

**Theorem 2:** There is a maximin solution to the endogenous sustainable
growth problem.

**Proof:** Define production in terms of effective capital and resource units.
That is, \( \frac{1}{L_{t}} F(L_{t}, Q_{t}, R_{t}, K_{t}) = f(k_{t}, q_{t}, r_{t}) \). Recall that compactness of the set of admissible programs depends on whether or not the model satisfies the technology conditions.

The arguments supporting technology conditions one and three follow directly. Assume the production function, \( f_{i} \), is concave in \( k_{t} \), \( q_{t} \) and \( r_{t} \). Investment is given by

\[
\dot{k}_{t} = f(k_{t}, q_{t}, r_{t}) - \theta_{q} k_{t} - c_{t}.
\]

Production is defined by

\[
G(t, k, q, r) = \{k_{t}, q_{t}, r_{t}\} = \left\{\begin{array}{c}
-\theta_{k} k_{t}, f(k_{t}, q_{t}, r_{t}) - \theta_{r} r_{t} \end{array}\right\}
\]

with \( r_{t} < r_{0} < X^{\text{max}} \) and \( q_{t} < q_{0} < W_{0} \).

It follows that

\[
\dot{k}_{t} \in \left[-\theta_{k} k_{t}, f(k_{t}, q_{t}, r_{t}) - \theta_{r} r_{t}\right],
\]

and,

\[
\Omega = \{(t, k, q, r, y) \in \mathbb{R}_{t}^{5} : y \in G(k_{t}, q_{t}, r_{t})\}.
\]

It is clear that since \( f \) is concave, \( G(\cdot) \) is compact and concave for each input. Therefore, the investment function, \( (k_{t}, q_{t}, r_{t}) \rightarrow G(k_{t}, q_{t}, r_{t}) \), is continuous provided \( c_{t} \) is continuous in \( k_{t}, q_{t}, \) and \( r_{t} \). Thus, upper semicontinuity is assured and the first technology condition is satisfied.

The third technology condition is trivially satisfied by the path of zero capital accumulation and zero extraction and harvest rates.

The second technology condition is satisfied for the natural resources since (i) the resources are assumed necessary for production and (ii) the nonrenewable resource is bounded above by \( W_{0} \), and the renewable resource follows a logistic
growth path, achieving a maximum sustainable population of \( X^{\text{max}} \) when \( S(X^{\text{max}}) = 0 \). \(^{11}\)

For effective physical capital,

\[
f(k_i, q_i, r_i) \leq f(k_0, W_0, X^{\text{max}}) + f_1(k_0, W_0, X^{\text{max}})(k_i - k_0),
\]

or,

\[
f(k_i, q_i, r_i) \leq f(k_0, W_0, X^{\text{max}}) - f_1(k_0, W_0, X^{\text{max}})(k_0 - k_i). \(^{12}\)
\]

If \( k_0 = 1 \), is the minimum initial stock of effective capital, then it follows that

\[
f(k_i, q_i, r_i) \leq f(1, W_0, X^{\text{max}}) - f_1(1, W_0, X^{\text{max}})(1 - k_i).
\]

Thus, \( f(k_i, q_i, r_i) \)

\[
\leq f(1, W_0, X^{\text{max}}) + f_1(1, W_0, X^{\text{max}})k_i - f_1(1, W_0, X^{\text{max}}) + f(1, W_0, X^{\text{max}})k_i + 2f_1(1, W_0, X^{\text{max}}).
\]

So,

\[
(4.1) \quad \dot{k}_i \leq f(k_i, q_i, r_i) \leq \left[ f(1, W_0, X^{\text{max}}) + f_1(1, W_0, X^{\text{max}}) \right](1 + k_i).
\]

Assuming continuity of \( f \), condition (4.1) and the Peano Existence Theorem ensure the existence of an optimal investment path. \(^{13}\) The upper bounds on the extraction/harvest rates ensure boundedness of the nonrenewable and renewable flow variables. Thus, the second technology condition is satisfied and existence is assured.

\[Q.E.D.\]

\(^{11}\)The "necessary" assumption is crucial, for it precludes complete exhaustion or extinction in finite time.

\(^{12}\)As expected, \( f_1 \) denotes the first partial derivative of the production function. In this application this is the marginal product of effective physical capital.

\(^{13}\)The Peano Existence Theorem gives sufficient criteria for the existence of a solution to the differential equation, \( \dot{k} = f(t, k) \) on \([0, \infty)\). The reasoning is that, assuming concavity of \( f \), there exist a continuous real valued function, \( \tau(t) \) such that \( 0 \leq f(k, \tau(t)) \leq \tau(t)(1 + k) \), \( k \geq 0 \). Let \( \tau_T = \sup \{ \tau(t) : 0 \leq t \leq T + 1 \} \). If there exists a function \( \tau \) such that \( f(k, \tau(t)) < t(1 + k) \), on any finite interval, then \( \dot{k} = f(t, k) \) exists on that interval. It is then possible to use the Peano Existence Theorem to ensure that the condition holds for some \( \tau^* \) on \([0, \infty)\). Thus, an investment plan exists on \([0, \infty)\) (Kolmogorov and Fomin (1975)).

In this application, \( \tau = \left[ f(1, W_0, X^{\text{max}}) + f_1(1, W_0, X^{\text{max}}) \right] \).
**Corollary:** There exists a maximin solution to the general problem of endogenous sustainable growth with heterogeneous physical capital and natural resources.

**Proof:** Refining the general framework and notation of section three, let 
\[ k = \left( k_{m_1}, \ldots, k_{m_{\alpha}}, k_{m_{\beta_1}}, \ldots, k_{m_{\beta_q}}, \ldots, k_{m_{\lambda_1}}, \ldots, k_{m_{\lambda_p}} \right) \in \mathbb{R}^n, \] where (effective physical) capital-consumption goods are given by \( k_i, i = m_{\alpha}, \ldots, m_{\alpha_p} \). The nonrenewable resource inputs are represented by \( k_i, i = m_{\beta_1}, \ldots, m_{\beta_q} \), and the renewable resource inputs are given by \( k_i, i = m_{\lambda_1}, \ldots, m_{\lambda_p} \).

A "reduced process" of a production set, \( \Pi \), is a pair, \((k(t), y(t))\) \( \in \mathbb{R}^n \times \mathbb{R}^m \), such that 
\[ y(t) = c(t) + \dot{k}(t). \] \(^{14}\)

Now, \( \Omega = \mathbb{R}^n \times \Pi \), with production, \( G(t, k) = \{ y \in \mathbb{R}^m : (k, y) \in \Pi \} \). It follows that \( G(t, k) \) is closed, and thus, compact valued (BB&S, p.94). Thus, the first technology condition is satisfied. As in theorem 2, the third technology condition is trivially satisfied. Since Peano's theorem is not restricted to differential equations in \( \mathbb{R}^2 \), it also applies, in a similar manner, to this more general situation. Thus, the third technology condition is satisfied, and existence of the maximin assured.

**Q.E.D.**

**5. Conclusion**

Using the basic framework of Becker, Boyd and Sung (1989), this analysis demonstrates existence of a maximin solution to this model of endogenous growth. This result is significant for two principal reasons. Most obviously it demonstrates that growth via the accumulation of human capital is compatible with the Rawlsian criterion for intergenerational equity, even when considering both renewable and nonrenewable resources. Second, this proof also applies to situations in which human capital is constant, and thus, it may be used as an alternative to the

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\(^{14}\)By Magill (1981): (i) \( \Pi \) is a closed, convex, cone, (ii) If \((k, y)\) \( \in \Pi \), and \( k = 0 \), then \( y = 0 \), (iii) There is a \((k, y)\) \( \in \Pi \) with \( y > 0 \), (iv) If \((k, y)\) \( \in \Pi \), \( k' \geq k \) and \( 0 \leq y' \leq y \), then \((k', y')\) \( \in \Pi \).
existence proof offered by Burmeister and Hammond (1977). In their version, Burmeister and Hammond employ a continuum of multipliers so as to provide a heuristic motivation for prices. In truth, general treatment of a continuum of constraints and multipliers remains a sophisticated, unsolved, mathematical problem. One thing is certain, the Lagrangian method cannot be arbitrarily applied to a system involving a continuum of nonlinear constraints. This fact calls into question the validity of Burmeister and Hammonds' widely cited result.

6. Appendix

The compactness argument is demonstrated herein. The first topology considered is the compact-open topology. The compact-open topology is defined by the family of norms, \( \|k\|_{k,T} = \sup_{0 \leq t \leq T} |k(t)| \), where \( |\cdot| \) is any of the equivalent \( \ell^p \) norms on \( R^m \). Under this topology, \( A \) is a complete, countably normed (Fréchet) space. This topology is used to insure compactness of the feasible set, \( U(x) \).

The second topology to be considered is the weak topology defined on \( A \). This topology is used to insure that the objective is upper semicontinuous. In order to properly define a weak topology on \( A \), BB&S first demonstrate that \( A \) is the direct sum \( R \oplus L^1_{loc} \). This is accomplished by defining a norm on \( A \) which uses the fact that the derivatives (input flows) of absolutely continuous functions (capital stock) are locally integrable, and that absolutely continuous functions can be described as the integral of its derivative. BB&S then show that the topology defined on \( A \) with respect to this norm is equivalent to the topology on \( R \oplus L^1_{loc} \) under the natural metric. Thus, \( A = R \oplus L^1_{loc} \). This makes \( A \) a Fréchet space, ensuring the existence of a dual space. The weak topology is defined by the dual space, the space of linear functionals on \( A \). By definition, the weak topology on \( A \) is the weakest topology which ensures that each linear functional of \( A \) is continuous. In order to ensure compactness of the feasible set, \( U(x) \), and upper semicontinuity of the objective simultaneously, it is essential that the two topologies be equivalent. This is achieved by focusing on the following subspace of \( A \), \( F(\eta) = \{ k \in A : \|k\| \leq \eta \} \) where \( \eta \in L^1_{loc} \). As a result, the two topologies are equivalent on the subspace consisting of those programs where the accumulation of physical capital is bounded by some \( \eta \in L^1_{loc} \).

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15 Recall that \( L^p = \left\{ k : \|k\|_p = \left( \sum_{i=1}^{\infty} |k_i|^p \right)^{1/p} < \infty \right\} \). See Packel (1974, p.55).
Lemma 1: On $\mathcal{F}(\eta)$, the compact-open and weak topologies are equivalent. (i) $\mathcal{F}(\eta)$ is compact in the compact-open topology. Moreover, $\mathcal{F}(\eta)$ is metrizable. (ii) $\mathcal{F}(\eta)$ is complete under the uniform metric.

Proof: (i) Becker, Boyd and Sung (1989, p.82-84).
(ii) Recall that each $k \in \mathcal{A}$ is absolutely continuous, with $k : \mathbb{R} \rightarrow \mathbb{R}^m$. Let $d$ be the usual metric on $\mathbb{R}^m$. The standard bounded metric on $\mathbb{R}^m$ is defined as $\bar{d}(x,y) = \min\{d(x,y),1\}$. Let $f, g$ be arbitrary functions from $\mathbb{R} \rightarrow \mathbb{R}^m$. The uniform metric for all such functions is given by $\rho(f,g) = \sup\{\bar{d}(f(t),g(t)) | t \in \mathbb{R}\}$. (In fact, $\rho(f,g) = \min\{\rho(f,g),1\}$ where $\rho(f,g) = \sup\{d(f(t),g(t)) | t \in \mathbb{R}\}$ is the sup metric.)

Let $\{k_n\} \subset \mathcal{F}(\eta)$ be a sequence of absolutely continuous functions converging to some $\tilde{k} : \mathbb{R} \rightarrow \mathbb{R}^m$ with respect to $\bar{\rho}$. We first show that this necessarily implies that the sequence, $\{k_n\} \subset \mathcal{F}(\eta)$ converges to $\tilde{k}$ uniformly (for $n \geq N$ for some $N$ and all $t \in \mathbb{R}$) relative to the metric $\bar{d}$ on $\mathbb{R}^m$.

Let $\varepsilon > 0$ be given. Choose an integer, $N$, such that $\bar{\rho}(k^n_1,k^n_n) < \varepsilon$ for all $n \geq N$. Then for all $t \in \mathbb{R}$ and all $n \geq N$, $\bar{d}(\tilde{k}(t),k_n(t)) \leq \bar{\rho}(\tilde{k},k^n_1) < \varepsilon$. Thus, $\{k_n\} \rightarrow \tilde{k}$ uniformly. By (i), $\mathcal{F}(\eta)$ is closed. Thus, $\tilde{k} \in \mathcal{F}(\eta)$ with $\|\tilde{k}\| < \eta$.

Recall that a topological space is metrizable if there exists a metric on the set $X$, which induces the topology of $X$. It follows that $\mathcal{U}(x) \subset \mathcal{F}(\eta)$. Since $\mathcal{F}(\eta)$ is compact, to show that the subset, $\mathcal{U}(x)$, is also compact it suffices to show that $\mathcal{U}(x)$ is closed with respect to the compact-open topology. In order to prove this BB&S combine two principle technical arguments. The first argument shows that if there exists a sequence, $\{k_n\}$, with $k_n \rightarrow k$ weakly in $L^1_{\text{loc}}$, for any $t$, $k_n$ is bounded below by the infimum of the sequence and bounded above by the supremum almost everywhere. The second condition shows that as a compact, convex, non-empty subset of $\mathbb{R}^{m+1}$, $G(t,k)$ can be expressed as the intersection of a countable number of half spaces that contain it. Assuming that $k_n \rightarrow k$, with each $k_n \in \mathcal{U}(x)$, it follows that $\mathcal{U}(x)$ is closed if $k(t) \in \mathcal{U}(x)$ for almost every $t$. The two conditions are employed in a way that essentially "traps" $\tilde{k}$ in $G(t,k)$ for
almost every \( t \). With \( \hat{k} \) in the production set, it follows that \( k \) is a feasible program. Thus, the following holds.

**Lemma 2:** If the technology conditions are satisfied, \( U(x) \subseteq F(\eta) \) is a compact subset of \( A \).

**Proof:** Becker, Boyd and Sung (1989, p.85-87).

**References**


