252power 2/8/01 **(Open this document in 'Page Layout' view!)**

C. POWER FUNCTIONS

1. A one-sided Test.

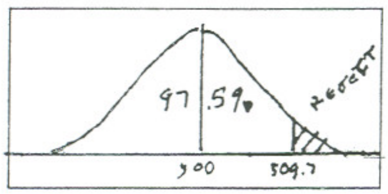
If we consider the problem: $\begin{cases} H_0 : \mu \le \mu_0 \\ H_1 : \mu > \mu_0 \end{cases}$ for a give value of α, to construct an operating characteristic curve, we must find the value of β for all possible values of μ_1 above μ_0. As a practical matter, let μ_1 equal μ_0, $\mu_0 + \frac{1}{2} z_\alpha \sigma_{\bar{x}}$, $\mu_0 + z_\alpha \sigma_{\bar{x}} \left(= \bar{x}_{cv}\right)$, $\mu_0 + \frac{3}{2} z_\alpha \sigma_{\bar{x}}$ and so on until β is almost one. The power function is a plot of $1-\beta$ instead of β, and crosses the operating characteristic curve at the critical value.

Example: Assume that $\alpha = .025$ and test $\begin{cases} H_0 : \mu \le 500 \\ H_1 : \mu > 500 \end{cases}$ when $\sigma = 35$ and $n = 50$, so that

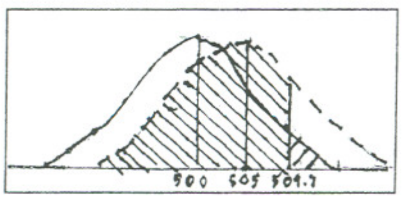
$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{35}{\sqrt{50}} = 4.95$ and $\bar{x}_{cv} = \mu_0 + z_\alpha \sigma_{\bar{x}} = 500 + (1.960)(4.95) = 509.7$. We thus reject the null hypothesis if the sample mean is above 509.7. Since the interval between μ_0 and $\bar{x}_{cv}$ is about 10, the interval between values of μ_1 should be about half that, so let μ_1 equal 505, 509.7, 515, and 520. (One value of μ_1 should be μ_0, and one should be $\bar{x}_{cv}$.)

Start then, by assuming that $\mu_1 = 505$. Then $\beta = P\left\{ Accepting'\, H_0 \middle| H_0 \text{ is false} \right\}$

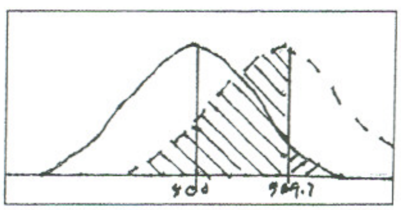
$$= P\left\{\bar{x} \le 509.7 \middle| \mu = 505\right\} = P\left\{ z \le \frac{\bar{x}_{cv} - \mu_1}{\sigma_{\bar{x}}} \right\} = P\left\{ z \le \frac{509.7 - 505}{4.95} \right\} = P\{z \le 0.95\} = .5 + .3289 = .8289$$



An illustration would help. Since our critical value is 509.7, we accept the null hypothesis only if the sample mean is less than or equal to 509.7. But let us assume that the population mean is actually 505. <u>In all the following diagrams, the 'reject' region is shaded Southwest to Northeast and the region representing the probability of wrongly failing to reject the null hypothesis is shaded Northwest to Southeast.</u>



Then we can draw two Normal curves, one centered at 505 and one centered at 500. The probability of rejecting the null hypothesis when it is true is the right shaded area, while the probability of accepting the null hypothesis when the mean is 505 is the left shaded area.

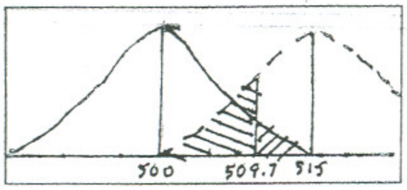


Now, try the second value, 509.7, for μ_1.

$\beta = P\left\{ Accepting'\, H_0 \middle| H_0 \text{ is false} \right\}$

$= P\left\{\bar{x} \le 509.7 \middle| \mu = 509.7\right\} = P\left\{ z \le \frac{\bar{x}_{cv} - \mu_1}{\sigma_{\bar{x}}} \right\}$

$= P\left\{ z \le \frac{509.7 - 509.7}{4.95} \right\} = P\{z \le 0\} = .5000$ Note that β at $\mu_1 = \bar{x}_{cv}$ is always .5000.



Next, lets try $\mu_1 = 515$.

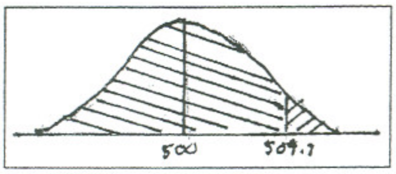
$$\beta = P\{\bar{x} \le 509.7 | \mu = 515\}$$

$$= P\left\{z \le \frac{509.7 - 515}{4.95}\right\} = P\{z \le -1.07\} = .1423.$$

For $\mu_1 = 520$.

$$\beta = P\{\bar{x} \le 509.7 | \mu = 520\}$$

$$= P\left\{z \le \frac{509.7 - 520}{4.95}\right\} = P\{z \le -2.08\} = .0188$$

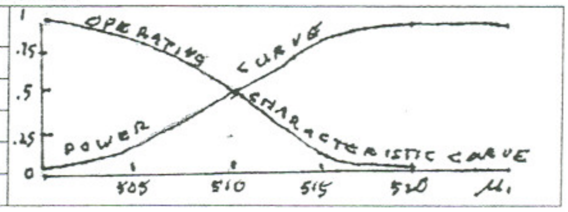


Finally, to get a limiting value, let $\mu_1 = 500$. $\beta = P\{\bar{x} \le 509.7 | \mu = 500\}$

$$= P\left\{z \le \frac{509.7 - 500}{4.95}\right\} = P\{z \le 1.96\} = .9750.$$ Note that β at $\mu_1 = \mu_0$ is always $1 - \alpha$.

We can summarize our results in the table and graph below.

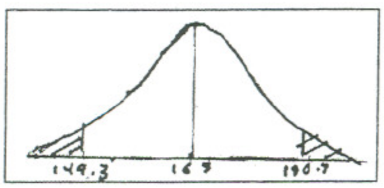
μ_1	β	Power $= 1 - \beta$
500	.9750	2.50%
505	.8289	17.11%
509.7	.5000	50.00%
515	.1423	85.77%
520	.0188	98.12%



2. A Two-Sided Test.

This is the same as a one sided test except that we must consider values of μ_1 on both sides of μ_0. This is little additional effort since points at the same distance from μ_0 have the same values of β.

<u>Example:</u> Assume that $\alpha = .05$ and test $\begin{cases} H_0 : \mu = 165 \\ H_1 : \mu \ne 165 \end{cases}$ when $\sigma = 48$ and $n = 36$, so that

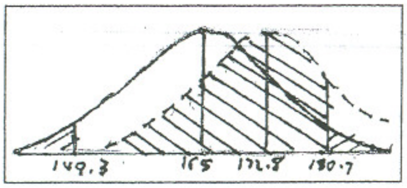


$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{48}{\sqrt{36}} = 8.00$$

$$\bar{x}_{cv} = \mu_0 \pm z_{\alpha/2} \sigma_{\bar{x}} = 165 + (1.960)(8.00) = 165 \pm 15.7$$

We thus reject the null hypothesis if the sample mean is below 149.3 or above 180.7.

Since the interval between μ_0 and $\bar{x}_{cv}$ is actually between 15.6 and 15.7, space μ_1 at half that, or about 7.8 or 7.9. The points that I picked above 165 were 172.8, 180.7, 188.5,and 196.4. The points below 165 were 157.2, 149.3, 141.5 and 133.6. Note that, for example 172.8 and 157.2 are at the same distance from 165.



Now, try 172.8 for μ_1. $\beta = P\{Accepting' H_0 | H_0 \text{ is false}\}$
$= P\{149.3 \le \bar{x} \le 180.7 | \mu = 172.8\}$
$= P\left\{\frac{149.3-172.8}{8} \le z \le \frac{180.7-172.8}{8}\right\} = P\{-2.94 \le z \le 0.99\}$
$= .4984 + .3389 = .8373$. Now try the same calculation for $\mu_1 = 157.7$, which is the same distance to the left of 165 as 172.8 is to the right.

This time
$\beta = P\{Accepting' H_0 | H_0 \text{ is false}\} = P\{149.3 \le \bar{x} \le 180.7 | \mu = 157.7\}$
$= P\left\{\frac{149.3-157.7}{8} \le z \le \frac{180.7-157.7}{8}\right\} = P\{-0.99 \le z \le 2.94\} = .3389 + .4984 = .8373$, which is the same result as for $\mu_1 = 172.8$. In other words the operating characteristics curve and power function curve for values of μ_1 to the left of μ_0 is the mirror image of the curve to the right of μ_0. The table below summarizes our results and further calculations. $\bar{x}_{cvL}$ means a lower critical value and $\bar{x}_{cvU}$ means an upper critical value. These become z_L and z_U. Note that, as above, when $\mu_1 = \mu_0$ Power $= \alpha$, and when $\mu_1 = \bar{x}_{cv}$ Power = 50%.

μ_1	$\frac{\bar{x}_{cvL} - \mu_1}{\sigma_{\bar{x}}}$	$\frac{\bar{x}_{cvU} - \mu_1}{\sigma_{\bar{x}}}$	z_L	z_U	β	Power $= 1-\beta$
165	$\frac{149.3-165}{8}$	$\frac{180.7-165}{8}$	-1.96	1.96	.4750 + .4750 = .9500	5.00%
172.8 (Same as 157.2)	$\frac{149.3-172.8}{8}$	$\frac{180.7-172.8}{8}$	-2.94	0.99	.4984 + .3389 = .8373	16.27%
180.7 (Same as 149.3)	$\frac{149.3-180.7}{8}$	$\frac{180.7-180.7}{8}$	-3.93	0.00	.5000	50.00%
188.5 (Same as 141.5)	$\frac{149.3-188.5}{8}$	$\frac{180.7-188.5}{8}$	-4.90	-0.98	.5000 − .3365 = .1635	83.65%
196.4 (Same as 133.8)	$\frac{149.3-196.4}{8}$	$\frac{180.7-196.4}{8}$	-5.98	-1.96	.5000 − .4750 = .0250	97.50%